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Linear Relaxation Times of Stochastic Processes Driven by Non-Gaussian Noises

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The linear relaxation time (LRT) associated with steady-state correlation functions is studied for Langevin equations with non-Gaussian noises: dichotomous Markov noise and Poissonian white shot noise. Exact results for arbitrary models are obtained and compared with results for Gaussian noises. Some general features of LRTs are discussed. The concept of dynamic effective diffusion is introduced and the existence of an optimal effective Fokker-Planck approximation is discussed. Explicit examples for prototype models are presented and briefly compared with the analogs for Gaussian noises.

KEY WORDS: Linear relaxation time; Gaussian white noise; white shot noise; dichotomous Markov noise; Ornstein-Uhlenbeck noise; dynamic (static) effective diffusion.

1. INTRODUCTION

Stochastic differential equations are widely used in the modeling of a great variety of nonequilibrium systems.[1-4] An interesting aspect from the point of view of the dynamics is the study of fluctuations in the steady state. In the framework of systems described by Langevin-like equations of the general form

$$\dot{x} = v(x) + g(x)\,\eta(t) \tag{1.1}$$

where $\eta(t)$ stands for a stochastic force or noise, the dynamics of fluctuations in the steady state is well characterized by the autocorrelation function of the variable x.[3,5-8] This quantity will have in general a complex dependence on both the deterministic forces $v(x)$ and on the statistical properties of the stochastic terms $g(x)\,\eta(t)$. The most usual assumption is

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to consider that $\eta(t)$ is a Gaussian white process. This is an idealization which has been very useful, but which may be too restrictive in the description of some real systems,[3,4] so other more general noises, non-Gaussian and/or nonwhite, should be taken into account. Nevertheless, for nonlinear problems no exact results for correlation functions are usually available, even for Gaussian white noise (GWN), so standard approximate techniques have been developed.[5–12]

Some recent results[10–12] concerning the so-called linear relaxation time (LRT) of correlation functions (defined as the time integral of the normalized correlation function) for GWN problems have given to this quantity a renewed interest. Here we generalize some of those techniques and obtain exact results concerning other types of noise.

The quantity we focus on in this paper is the so-called linear relaxation time[13,14] (LRT) (which has also been referred to as the relaxation time, the correlation time, the mean relaxation time, etc.). This quantity has often been taken into account in the literature as a global characterization of the decay of correlation functions[8,13,14] and it has been interpreted as a typical time scale of the relaxation of fluctuations in the steady state. The qualificative of "linear" is to distinguish it from its counterpart, the nonlinear relaxation time, which is the analog for the transient evolution.[14–16] The former makes reference to an eventually "linear" relaxation of small fluctuations around the steady state, in opposition to the relaxation of an initial condition which, being in general far form the steady state, involves nonlinear dynamics.

The LRT of a steady-state correlation function of the general form

$$C_{12}(s) = \lim_{t \to \infty} \langle f_1(x(t+s))\, f_2(x(t))\rangle - \langle f_1 \rangle_{st} \langle f_2 \rangle_{st} \tag{1.2}$$

is defined by

$$T_{12} = \frac{1}{C_{12}(0)} \int_0^\infty C_{12}(s)\, ds \tag{1.3}$$

The idea of the definition (1.3) is that, in an expansion of the form

$$C_{12}(s) \sim \sum_k a_k e^{-\mu_k s} \tag{1.4}$$

it can be seen as an average with the weights a_k of the relaxation times μ_k^{-1} of the exponentials (mean relaxation time[10,11]). In this sense, this would be the complementary definition of that of the so-called effective eigenvalue[17] which corresponds, in an expansion of the form (1.4), to the average of the eigenvalues μ_k. Another interpretation of both characteristic times was pointed out by Nadler and Schulten[10,11] in a more general framework as the coefficients of the lowest orders in low-frequency and high-frequency expansions, respectively. Of concern to us here, their main result is that the LRT (and the so-called low-frequency moments) in the case of GWN can be exactly calculated without the explicit knowledge of the correlation function. The same result for the LRT was obtained by Jung and Risken.[12]

In this paper we present a generalization of the method followed by Jung and Risken[12] for the calculation of the LRT, which makes it possible to obtain the hierarchy of low-frequency moments of Nadler and Schulten.[10,11] The problem is set up in a simple mathematical framework which applies also to noises other than Gaussian and white. Our concern here will be essentially on the lowest order, the LRT. An approximate extension to systems driven by Gaussian colored noise was developed in ref. 18. In this paper we apply the theory to some exactly solvable cases involving non-Gaussian noises.

The general solution of the LRT associated with a correlation function (1.2) defined by an equation (1.1) and a given noise $\eta(t)$ consists of a functional of the quantities $v(x)$, $g(x)$, $f_1(x)$, $f_2(x)$, and the parameters of the noise. Given that in general the correlation function is not exactly known, the exact knowledge of the LRT (and the successive low-frequency moments) can be interesting for different reasons. First, from the analysis of the functional form of the LRT one can obtain general information about the specific features of dynamics under the influence of any particular noise, as we will see, for instance, in the discussion on the existence of effective diffusion functions in Section 3.1. Second, the analysis of the solution for particular choices of $v(x)$, $g(x)$, $f_1(x)$, and $f_2(x)$ for prototype models can be useful as a dynamical characterization of different physical situations. We include a brief discussion of some illustrative examples in Section 4. An explicit evaluation of the LRT can also be useful as a test for the different approximate methods in particular situations.[19] Finally an interesting point is that the information contained in the LRT (and the low-frequency moments in general) can be used in a systematic way in order to get explicit approximations of the correlation function itself as proposed in refs. 10, 11, and 19.

2. EXACT RESULTS

2.1. Formulation of the Problem

In this section we generalize the method followed in ref. 12 for the calculation of the LRT of a steady-state correlation function in a GWN single-variable problem. The generalized form applies to other noises and multivariable systems and reformulates in a simple way the problem of the calculation of the low-frequency moments of refs. 10, 11, and 19. The starting point is the evolution equation of the steady-state two-time ($t - t' = s$) joint probability density $P_{st}(x, x'; s)$ as the quantity which generates all the steady-state correlation functions,

$$\frac{\partial}{\partial s} P_{st}(x, x'; s) = L(x)\, P_{st}(x, x'; s) \tag{2.1}$$

In Eq. (2.1) it is implicitly assumed that the process $x(t)$ is Markovian. Nevertheless, the formalism is also valid for multivariable systems, so non-Markovian processes can be incorporated when reformulated in its equivalent multivariable Markovian formulation.

According to (2.1) and taking into account the initial condition

$$P_{st}(x, x'; 0) = \delta(x - x')\, P_{st}(x') \tag{2.2}$$

one can write the correlation function $C_{12}(s)$ as

$$C_{12}(s) = \int_a^b dx\, f_1(x)\, e^{L(x)s}[f_2(x) - \langle f_2 \rangle_{st}]\, P_{st}(x) \tag{2.3}$$

where (a, b) is the natural domain of the process $x(t)$. We define now the quantity

$$W(x, s) = e^{L(x)s}[f_2(x) - \langle f_2 \rangle_{st}]\, P_{st}(x) \tag{2.4}$$

and its Laplace transform

$$\rho(x, w) = \int_0^\infty e^{-ws}\, W(x, s)\, ds \tag{2.5}$$

Notice that $W(x, s)$ obeys, from its definition, the equation

$$\frac{\partial}{\partial s} W(x, s) = L(x)\, W(x, s) \tag{2.6}$$

so, taking into account that $W(x, \infty)$ must vanish and $W(x, 0) = [f_2(x) - \langle f_2 \rangle_{st}]\, P_{st}(x)$, the Laplace transform of (2.6) reads

$$-[f_2(x) - \langle f_2 \rangle_{st}]\, P_{st}(x) = [L(x) - w]\, \rho(x, w) \tag{2.7}$$

On the other hand, the Laplace transform of the $C_{12}(s)$ using (2.5) will be given by

$$\hat{C}_{12}(w) = \int_a^b f_1(x)\, \rho(x, w)\, dx \tag{2.8}$$

where we have commuted the order of the integrations.

Thus, the problem of determining the $\hat{C}_{12}(w)$ has been formulated in terms of the differential equation (2.7). If it is possible to obtain $\rho(x, w)$ from (2.7), and this is the key point of the method, we will have reduced the problem to quadrature by simply inserting the $\rho(x, w)$ into (2.8).

Provided a direct solution of (2.7) is not generally possible, in order to get a low-frequency characterization of $\hat{C}_{12}(w)$ we will assume an expansion of $\rho(x, w)$ in powers of w of the form

$$\rho(x, w) = \rho_0(x) + w\rho_1(x) + w^2\rho_2(x) + \cdots \tag{2.9}$$

so that Eq. (2.7) reduces to the infinite set of equations

$$L(x)\, \rho_0(x) = -[f_2(x) - \langle f_2 \rangle_{\text{st}}]\, P_{\text{st}}(x) \tag{2.10}$$

$$L(x)\, \rho_n(x) = \rho_{n-1}(x); \qquad n \geqslant 1 \tag{2.11}$$

The successive orders can be solved recursively provided the lowest order, which corresponds to the LRT, is solvable. Assuming the corresponding w expansion of the $\hat{C}_{12}(w)$

$$\hat{C}_{12}(w) = C_{12}^{(0)} + C_{12}^{(1)}w + C_{12}^{(2)}w^2 + \cdots \tag{2.12}$$

the coefficients $C_{12}^{(n)}$ (essentially the low-frequency moments of refs. 10 and 11), and in particular the LRT T_{12}, will be given by

$$T_{12} = C_{12}^{(0)}/C_{12}(0); \qquad C_{12}(0) = \langle f_1 f_2 \rangle_{12} - \langle f_1 \rangle_{\text{st}} \langle f_2 \rangle_{\text{st}} \tag{2.13}$$

$$C_{12}^{(n)} \equiv (-1)^n \frac{1}{n!} \int_0^\infty s^n C_{12}(s)\, ds = \int_a^b f_1(x)\, \rho_n(x)\, dx \tag{2.14}$$

In our formulation, there are no strong restrictions on the form of $L(x)$, so, in principle, a wide range of different possibilities can be admitted. The practical usefulness of the method in a particular case lies then in two steps: the knowledge of the $L(x)$ and the solvability of Eq. (2.10). Notice, however, that Eq. (2.10) is the one obeyed by the stationary probability density $P_{\text{st}}(x)$ of the system except for an inhomogeneous term. In this case it has to be solved with the usual condition of vanishing current at the natural boundaries and subject to the additional conditions

$$\int_a^b \rho_n(x)\,dx = 0; \qquad n \geqslant 0 \tag{2.15}$$

which arise from the fact that the integral over the domain (a, b) is conserved under the evolution generated by the operator $\exp[L(x)s]$, so that

$$\int_a^b W(x, s)\,dx = 0, \qquad \forall s \tag{2.16}$$

From a practical point of view, this implies that for one-variable problems (2.10) will be exactly solvable in those cases for which the $P_{\rm st}(x)$ is known, and, similarly, if $P_{\rm st}(x)$ is only approximately known, as for Gaussian colored noise, the LRT will be solvable in the same approximation.[18]

In this paper we present some exactly solvable cases by means of this procedure, including both Markovian and non-Markovian cases. For further reference we include here the general solution for the GWN case.[10–12] The Stratonovich interpretation of (1.1) leads to a Fokker–Planck operator of the form

$$L(x) = -\frac{\partial}{\partial x} v(x) + D \frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g(x) \tag{2.17}$$

and a straightforward application of the method to it gives

$$T_{12} = \frac{1}{\langle f_1 f_2 \rangle_{\rm st} - \langle f_1 \rangle_{\rm st} \langle f_2 \rangle_{\rm st}} \int_a^b \frac{F_1(x)\, F_2(x)}{D g^2(x)\, P_{\rm st}(x)}\, dx \tag{2.18a}$$

$$F_i(x) \equiv -\int_a^x [f_i(x') - \langle f_i \rangle_{\rm st}]\, P_{\rm st}(x')\, dx' \tag{2.18b}$$

as the solution for GWN in the most general case.

2.2. Dichotomous Markov Noise (DMN)

This section is devoted to the case in which the noise $\eta(t)$ in (1.1) is a dichotomous Markov process (DMN),[4,20–22] which takes the values Δ and Δ'. We denote by μ and μ' the respective transition rates between these two states of the noise. The condition of vanishing mean value reads

$$\frac{\Delta}{\mu} + \frac{\Delta'}{\mu'} = 0 \tag{2.19}$$

The noise is thus characterized by three independent parameters. The correlation function of the noise is

$$\langle \eta(t)\,\eta(t')\rangle = \frac{\mu\mu'}{(\mu+\mu')^2}(\Delta-\Delta')^2 \exp[-(\mu+\mu')\,|t-t'|] \qquad (2.20)$$

It has a finite correlation time $\tau=(\mu+\mu')^{-1}\equiv\Lambda^{-1}$, and hence the process $x(t)$ defined by (1.1) is non-Markovian.

An important particular case is the symmetric dichotomous noise (SDMN), which takes the values $\pm\Delta$ and with $\mu=\mu'$. This case has a GWN limit when $\tau\to 0$ in such a way that $\Delta^2\tau=D=\text{const}$.

Although the process $x(t)$ is non-Markovian, the procedure of Section 2.1 will apply to the augmented two-variable problem (x, η), which is then Markovian. The discrete character of the noise variable is crucial for our purposes because it permits the formulation of the problem in terms of a system of ordinary differential equations [instead of a partial differential equation, as would be the case for an Ornstein–Uhlenbeck noise (OUN) problem].

By choosing the appropriate auxiliary quantities as the second components,[21] we can define the vector $\mathbf{P}_{st}(x, x'; s)$ which obeys the equation

$$\frac{\partial}{\partial s}\mathbf{P}(x, x'; s) = \hat{L}(x)\,\mathbf{P}_{st}(x, x'; s) \qquad (2.21)$$

where the operator $\hat{L}(x)$ is the matrix

$$\begin{pmatrix} -\dfrac{\partial}{\partial x}\left[v(x)+\dfrac{\Delta+\Delta'}{2}g(x)\right] & \dfrac{\Delta-\Delta'}{2}\dfrac{\partial}{\partial x}g(x) \\ \mu-\mu'+\dfrac{\Delta-\Delta'}{2}\dfrac{\partial}{\partial x}g(x) & -\left\{\mu+\mu'+\dfrac{\partial}{\partial x}\left[v(x)+\dfrac{\Delta+\Delta'}{2}g(x)\right]\right\} \end{pmatrix} \qquad (2.22)$$

and the quantity $P_{st}(x, x'; s)$ in which we are interested is the first component of the vector $\mathbf{P}_{st}(x, x'; s)$. In this formulation we will also have a vector correlation function

$$\mathbf{C}_{12}(s) = \int_a^b dx\, f_1(x)\{\exp[\hat{L}(x)s]\}[f_2(x)-\langle f_2\rangle_{st}]\,\mathbf{P}_{st}(x)\,dx \qquad (2.23)$$

whose first component is the nonnormalized correlation function associated with the non-Markovian process $x(t)$ defined by (1.2).

The same formal scheme as in Section 2.1 can be followed now. The equation to solve takes the form

$$\hat{L}(x)\, \boldsymbol{\rho}_0(x) = -[f_2(x) - \langle f_2 \rangle_{st}]\ \mathbf{P}_{st}(x) \tag{2.24}$$

and we are interested in the first component of $\boldsymbol{\rho}_0(x)$. After some algebra and using relations between the two components of $\mathbf{P}_{st}(x)$,[20] the solution in the case $f_1 = f_2 = f$ can be arranged as

$$T = \frac{1}{C(0)} \int_a^b \frac{F^2(x)}{D^s_{\text{eff}}(x)\, P_{st}(x)} \times \left\{ \Delta + \left[v(x) + \frac{\Delta + \Delta'}{2}\, g(x) \right] \frac{d}{dx} \ln \frac{D^s_{\text{eff}}(x)\, P_{st}(x)}{g(x)} \right\} dx \tag{2.25}$$

where

$$F(x) = -\int_a^x [f(x') - \langle f \rangle_{st}]\, P_{st}(x')\, dx' \tag{2.26}$$

and $D^s_{\text{eff}}(x)$ is what we will call the static effective diffusion (see Section 3.1),

$$D^s_{\text{eff}}(x) = -[v(x) + \Delta g(x)][v(x) + \Delta' g(x)] \tag{2.27}$$

This form is very convenient, since taking into account the explicit form of the $P_{st}(x)$

$$P_{st}(x) \propto \frac{g(x)}{D^s_{\text{eff}}(x)} \exp\left\{ \Delta \int^x \frac{v(x')}{D^s_{\text{eff}}(x')}\, dx' \right\} \tag{2.28}$$

Eq. (2.25) reduces to

$$T = \frac{\Delta}{C(0)} \int_a^b \frac{F^2(x)}{D^s_{\text{eff}}(x)\, P_{st}(x)} \left\{ 1 + \frac{v(x)}{D^s_{\text{eff}}(x)} \left[v(x) + \frac{\Delta + \Delta'}{2}\, g(x) \right] \right\} dx \tag{2.29}$$

This result is exact and refers to the general LRT problem of the process $x(t)$, which is non-Markovian. Equation (2.29) may be written in a more compact form as

$$T = \frac{1}{C(0)} \int_a^b \frac{F^2(x)}{D_{\text{eff}}(x)\, P_{st}(x)}\, dx \tag{2.30}$$

The formal comparison with the GWN solution (2.18) suggests the definition of what we will call a dynamic effective diffusion $D_{\text{eff}}(x)$, that is, the function appearing instead of $Dg^2(x)$,

$$D_{\text{eff}}(x) = -\frac{1}{\Delta} \frac{[v(x) + \Delta g(x)]^2 [v(x) + \Delta' g(x)]^2}{\Delta\Delta' g^2(x) + \frac{1}{2}(\Delta + \Delta')\, g(x)\, v(x)} \tag{2.31}$$

$$= D^s_{\text{eff}}(x) \frac{1}{\Delta} \frac{[v(x) + \Delta g(x)][v(x) + \Delta' g(x)]}{\Delta\Delta' g^2(x) + \frac{1}{2}(\Delta + \Delta')\, g(x)\, v(x)} \tag{2.32}$$

We will comment on this point in Section 3.1.

2.3. Poissonian White Shot Noise (WSN)

In this section we study processes driven by Poissonian white shot noise.[23,24] This is a process defined as the sum

$$\bar{\eta}(t) = \sum_i w_i\, \delta(t - t_i) \tag{2.33}$$

where t_i are random time points distributed with a given average time spacing λ^{-1}, so that the probability to have n such time points in a time interval of duration t is given by the Poisson distribution

$$P_n(t) = \frac{1}{n!} (\lambda t)^n e^{-\lambda t} \tag{2.34}$$

The δ pulses are weighted by w_i, which are random independent variables with a probability density $\varphi(w)$. Here we will consider the case in which $\varphi(w)$ is exponentially distributed as

$$\varphi(w) = \frac{1}{w_0} e^{-w/w_0} \theta(w); \qquad w_0 > 0 \tag{2.35}$$

where $\theta(w)$ stands for the Heaviside function. The process $\bar{\eta}(t)$ is white, but it is non-Gaussian, since its cumulants are all nonvanishing, though δ-correlated.[23,24] In order to have zero mean value, we redefine it as

$$\eta(t) = \bar{\eta}(t) - \lambda w_0 \tag{2.36}$$

Now the process $\eta(t)$ has a Gaussian limit given by $w_0 \to 0$, $\lambda \to \infty$, $\lambda w_0^2 = D = \text{const}$.

The probability density $P(x, t)$ of the process $x(t)$ defined by the

Langevin equation (1.1) with this WSN (in Stratonovich interpretation) obeys the equation[23]

$$\frac{\partial}{\partial t} P(x, t) = \left\{ -\frac{\partial}{\partial x} [v(x) - \lambda w_0 g(x)] - \lambda w_0 \frac{\partial}{\partial x} g(x) \left[1 - w_0 \frac{\partial}{\partial x} g(x) \right]^{-1} \right\} P(x, t) \tag{2.37}$$

The WSN itself can also be obtained from the asymmetric DMN taking the limits $\mu' \to \infty$, $\Delta' \to \infty$, with $\Delta'/\mu' = w_0 = \text{const}$. The parameter μ then plays the role of λ, and Δ becomes $-\lambda w_0$, which is the constant value that compensates in the average the δ pulses. The general solution of the LRT can then be obtained from that of the DMN in this limit. Nevertheless, as a more explicit illustration of the method of Section 2.1, we apply it in detail to this case with the operator of (2.37) in the Appendix.

The general result (A.10) can also be written in the compact form (2.30), which defines the corresponding dynamic effective diffusion as

$$D_{\text{eff}}(x) = 2w_0 g(x) \frac{[\lambda w_0 g(x) - v(x)]^2}{2\lambda w_0 g(x) - v(x)} \tag{2.38}$$

or, in terms of the parameters $D = \lambda w_0^2$ and w_0,

$$D_{\text{eff}}(x) = Dg^2(x) \frac{[1 - (w_0/D)\, v(x)/g(x)]^2}{1 - (w_0/2D)\, v(x)/g(x)} \tag{2.39}$$

3. SOME GENERAL REMARKS ON LRTs

3.1. Dynamic Effective Diffusion Function

An interesting aspect in the discussion of the results of Section 2 is the occurrence of what we called the dynamic effective diffusion. That function arises when considering the formal analogy of the general solution of the LRT for any autocorrelation function in the exactly solved cases.

For GWN, the general solution for a generic correlation function can be seen essentially as a functional of two objects: the steady-state probability density $P_{\text{st}}(x)$ and the diffusion function $Dg^2(x)$. It is a well-known fact that the dynamics associated with a given stationary solution $P_{\text{st}}(x)$ is not unique, so the explicit appearance of the diffusion function in the expression of the LRT fixes, in a sense, the dynamics of our problem. Therefore, an important part of the dynamic information is contained in this state-dependent diffusion coefficient. With this in mind, it is quite remarkable that for systems which do not obbey a Fokker–Planck equation, there exists a function playing the role of an effective diffusion function generating all the LRTs and that we can easily interpret in a Fokker–Planck scenario. The fact that the steady-state dynamics can be reduced exactly, at least as far as LRTs are concerned, to so simple a Markovian description is a remarkable point which is not guaranteed at all. For instance, such a dynamic effective diffusion does not exist for an Ornstein–Uhlenbeck noise (OUN) problem, even to first order in τ[18]

The knowledge of the dynamic effective diffusion can given a valuable understanding of the specific features of the steady-state dynamics associated with each particular noise, but the explicit dependence on $v(x)$, $g(x)$, and the parameters of the noise is not radically different from that of the static effective diffusion. This last quantity, also referred to in the preceding section, would be defined through a formal analogy between the steady-state probability density $P_{st}(x)$ and that corresponding to the GWN case, as the quantity appearing instead of $Dg^2(x)$. In fact, one usually can write the $P_{st}(x)$ as

$$P_{st}(x) = \frac{N}{g(x)\,\phi(x)} \exp \int^x \frac{v(x')}{Dg^2(x')\,\phi(x')}\,dx' \tag{3.1}$$

which defines the static effective diffusion as $D^s_{eff}(x) = Dg^2(x)\,\phi(x)$. In our results, the dynamic effective diffusion has turned out to be a multiple of the static one, so that we can write $D_{eff}(x) = Dg^2(x)\,\phi(x)\,\psi(x)$. However, the theoretical interest of the dynamic effective diffusion is that it has been defined from dynamic arguments. In the sense we commented on for GWN, it "fixes" the dynamics of an effective Markovian process which not only contains the exact statics $[P_{st}(x)]$, but also some exact dynamical information (all the exact LRTs). So the existence of a dynamic effective diffusion relates us immediately to an optimal Fokker–Planck approximation of the problem that would be defined by the Fokker–Planck operator

$$\begin{aligned} L(x) = &-\frac{\partial}{\partial x}\,[v(x)\,\psi(x) + Dg^2(x)\,\phi(x)\,\psi'(x)] \\ &+ D\,\frac{\partial}{\partial x}\,g(x)\,\frac{\partial}{\partial x}\,g(x)\,\phi(x)\,\psi(x) \end{aligned} \tag{3.2}$$

The reduction to this framework can also be useful for practical purposes of calculation or simulation.

3.2. Autocorrelation of the Drift

Here we will consider a particular choice of $f(x)$ for which the general solution of the LRT takes a very simple form with arbitrary $v(x)$ and $g(x)$ and for any noise among those studied in Sections 3.1–3.3.

Let us consider first the Gaussian white noise case. From the form of the general solutions of the LRT it is clear that the choice $f(x) = v(x)/g(x)$ makes the integral (2.17) immediate, so that it reduces to an average form

$$T_{v/g} = \frac{-1}{\langle g(v/g)' \rangle_{\text{st}}} \tag{3.3}$$

The quantity v/g has a simple interpretation. It is nothing but the drift of a Langevin equation related to (1.1) by a change of variable x and for which the noise would appear additively.

The remarkable feature of LRTs we have just pointed out for GWN is not restricted to this particular case. From the knowledge of the explicit form of the $P_{\text{st}}(x)$ in each case it is easy to show that in the cases studied in this paper the LRT for the quantity v/g reduces always to an average form. In fact, in the notation of Section 3.1 for the static and dynamic effective diffusion functions, $Dg^2(x)\,\phi(x)$ and $Dg^2(x)\,\phi(x)\,\psi(x)$, respectively, the LRT for the quantity v/g reads

$$T_{v/g} = -\frac{\langle \phi/\psi \rangle_{\text{st}}}{\langle \phi g(v/g)' \rangle_{\text{st}}} = \frac{D\langle \phi/\psi \rangle_{\text{st}}}{\langle (v/g)^2 \rangle_{\text{st}}} \tag{3.4}$$

taking into account that $F = DgP_{\text{st}}$ and $C(0) = \langle (v/g)^2 \rangle_{\text{st}} = -D\langle \phi g(v/g)' \rangle_{\text{st}}$. Equation (3.4) is exact for WSN and DMN with the correspondences of Table I. For the OUN case it is only approximate. Actually, a dynamic effective diffusion in the sense discussed in Section 3.1 does not exist in this case,[18] but we can write

$$T_{v/g} = -\frac{\langle \phi \rangle_{\text{st}}}{\langle \phi g(v/g)' \rangle_{\text{st}}} + \tau + O(\tau^2) \tag{3.5}$$

with $\phi(x)$ given by Table I.[25]

We include these expressions for their practical usefulness, for instance, in approximate calculations like those involved in perturbative expansions on D of the LRTs (see Section 4).

A simple and interesting application of these considerations is the class of multiplicative noise models of the form

$$\dot{x} = \frac{1}{n-1}\,[x - x^n + x\eta(t)]; \qquad n \geqslant 2 \tag{3.6}$$

Table I

	$\phi(x)$	$\psi(x)$
Dichotomous Markov noise (DMN) $(D=-\Delta\Delta'\tau)$	$\frac{1}{\Delta\Delta' g^2}(v+\Delta g)(v+\Delta' g)$	$\frac{(v+\Delta g)(v+\Delta' g)}{\Delta\Delta' g^2+\frac{1}{2}(\Delta+\Delta')\, gv}$
Symmetric Dichotomous Markov noise (SDMN) $(D=\Delta^2\tau)$	$1-\frac{\tau}{D}\left(\frac{v}{g}\right)^2$	$1-\frac{\tau}{D}\left(\frac{v}{g}\right)^2$
Poissonian white shot noise (WSN) $(D=\lambda w_0^2)$	$1-\frac{w_0}{D}\frac{v}{g}$	$\frac{1-(w_0/D)v/g}{1-\frac{1}{2}(w_0/D)\, v/g}$
Ornstein–Uhlenbeck noise (OUN)	$1+\tau g\left(\frac{v}{g}\right)'+O(\tau^2)$	

which are related to the Verhulst model ($n=2$) through the change $x \to x^{1/(n-1)}$. We know, from the general condition $\langle v/g\rangle_{st}=0$, that for this model $\langle x\rangle_{st}=1$ (actually this is valid for any noise with zero mean), so the autocorrelation of $v(x)/g(x)=1-x$ is precisely the usual correlation function $[f(x)=x-\langle x\rangle_{st}]$. For GWN it turns out that $T=T_{v/g}=1/\langle x\rangle_{st}=1$ (independent of D). This is a well-known result,[12] which is reobtained here in a very simple way. Similarly, for each n in (3.6) there is a mode x^{n-1} whose autocorrelation has a constant LRT which is exactly $T=1$.

As a direct consequence of (3.4), the LRT of the correlation function for the Verhulst model $[f(x)=x-1]$ with SDMN[26] reduces to the calculation of mere steady-state moments. The exact result then reads

$$T=\frac{\Delta^2\tau}{\langle x^2\rangle_{st}-1} \tag{3.7}$$

where the second moment is exactly known in terms of hypergeometric functions and reads

$$\langle x^2\rangle_{st}=4\,\frac{A+\Delta^2-1}{A+2\Delta-1}\,\frac{F(B,C;B+1;k)}{F(B,C;B-1;k)};\qquad \Delta>1 \tag{3.8a}$$

$$\langle x^2\rangle_{st}=(\Delta+1)^2\,\frac{F(-A-1,C;-A;1/k)}{F(1-A,C;-A;1/k)};\qquad \Delta<1 \tag{3.8b}$$

with

$$A=\frac{\Delta}{\Delta^2-1};\qquad B=\frac{\Delta}{2(\Delta-1)}+1;\qquad C=\frac{\Delta}{2(\Delta+1)};\qquad k=\frac{\Delta+1}{2\Delta} \tag{3.8c}$$

To first order in τ, the same model but with OUN gives rise to

$$T = 1 + \tau\langle x^2\rangle_0 + O(\tau^2) \tag{3.9}$$

where the $\langle\cdots\rangle_0$ (GWN average) is given by

$$\langle x^n\rangle_0 = D^n \frac{\Gamma(1/D+n)}{\Gamma(1/D)}; \qquad \langle x^2\rangle_0 = 1 + D \tag{3.10}$$

In both (3.7) and (3.9) it is clear that the color of the noise breaks down the property of independence of the LRT on the noise intensity D.

For completeness we also write the LRT of the Verhulst model with WSN, which reads

$$T = \frac{\lambda w_0^2}{\langle x^2\rangle_{\rm st} - 1} \tag{3.11}$$

4. ASYMPTOTIC RESULTS. SOME EXAMPLES

Here we briefly discuss some illustrative examples of asymptotic laws obtained for LRTs in different prototype situations, which are not intended to be exhaustive.

As a representative model we will consider that defined by (1.1) with

$$v(x) = -(\alpha x + \beta x^2 + \gamma x^3); \qquad \gamma > 0; \quad g(x) = 1 \tag{4.1}$$

We will call the case $\alpha > 0$, $\beta^2/\gamma < 4$ the monostable model. The bistable model will be defined by $\alpha < 0$, $\beta = 0$. The case $\alpha = \beta = 0$ is the so-called marginally stable model. Unless we indicate the contrary, in the following we will only consider the LRT associated with the autocorrelation of the variable x $[f(x) = x]$.

For Gaussian white noise (GWN), a perturbative expansion on the intensity of the noise for the monostable model leads to

$$T = \frac{1}{\alpha} + D\frac{1}{\alpha^4}(5\beta^2 - 3\alpha\gamma) + O(D^2) \tag{4.2}$$

This perturbative approach fails when one approaches the marginal point ($\alpha^2/\gamma \ll D$). For the marginal case $\alpha = \beta = 0$ the LRT can be evaluated exactly and reads

$$T = (\gamma D)^{-1/2}\delta \tag{4.3}$$

where $\delta = 0.7212....$ Notice that, in contrast to what the linear analysis of (4.2) suggests, for a finite intensity of the noise the LRT is always finite and only get a divergence in the deterministic limit $D \to 0$ (asymptotic critical slowing down[27]). For multiplicative noise models, instead, the LRT may present a divergence even for a finite intensity of noise.[19]

For the bistable model in the same limit of weak noise, a steepest descent analysis of the general result gives

$$T \sim \frac{\pi}{\alpha\sqrt{2}} e^{\alpha^2/4\gamma D} \tag{4.4}$$

This coincides with the inverse of the first nonvanishing eigenvalue of the problem in the same approximation, which dominates the passage through the barrier. Other subdominant time scales, such as that of the relaxation inside either of two wells, can be obtained by studying the autocorrelation of x^2 (see a discussion of this point in ref. 11).

For the case of white shot noise (WSN), a simple law can be obtained in the monostable case following a similar perturbative analysis now on the parameter w_0. To first order, we get a correction associated with the intrinsic asymmetry of WSN which reads

$$T = \frac{1}{\alpha} - w_0 \frac{\beta}{\alpha^2} + O(w_0^2) \tag{4.5}$$

If the noise is close to the Gaussian limit (w_0 and $D = \lambda w_0^2$ of comparable size), then we can write

$$\begin{aligned} T &= \frac{1}{\alpha} - w_0 \frac{\beta}{\alpha^2} + D \frac{5\beta^2 - 3\alpha\gamma}{\alpha^4} + O(w_0^2, w_0 D) \\ &= T_{\text{GWN}}(D) - w_0 \frac{\beta}{\alpha^2} + O(w_0^2, w_0 D) \end{aligned} \tag{4.6}$$

where $O(w_0^2)$ refers to non-Gaussian contributions.

For the case of colored noises it is also possible to obtain simple laws for the monostable model by means of perturbative expansions on the noise intensity D of the form

$$T = T_0(\tau) + D T_1(\tau) + D^2 T_2(\tau) + \cdots \tag{4.7}$$

The part $T_0(\tau)$ is common for symmetric dichotomous noise (SDMN) and Ornstein–Uhlenbeck noise (OUN) and reads exactly[28]

$$T_0(\tau) = \frac{1}{\alpha} + \tau \tag{4.8}$$

The contribution $T_1(\tau)$ instead is model dependent and reads, respectively, for SDMN and OUN

$$T_1^{\mathrm{SD}}(\tau)=\frac{5\gamma}{\alpha^3}\left[\frac{\beta^2}{\alpha\gamma}-\frac{3}{5}+\alpha\tau\left(1-\frac{2\beta^2}{5\alpha\gamma}\right)\right]+O(\tau^2) \tag{4.9a}$$

$$T_1^{\mathrm{OU}}(\tau)=\frac{5\gamma}{\alpha^3}\left[\frac{\beta^2}{\alpha\gamma}-\frac{3}{5}+\alpha\tau\left(\frac{3}{5}-\frac{\beta^2}{3\alpha\gamma}\right)\right]+O(\tau^2) \tag{4.9b}$$

From this result we can conclude, for instance, that for the monostable model

$$T^{\mathrm{SD}}-T^{\mathrm{OU}}=D\tau\frac{\gamma}{3\alpha^2}\left(6-\frac{\beta^2}{\alpha\gamma}\right)+O(D\tau^2, D^2\tau)>0 \tag{4.10}$$

Finally, the steepest descent analysis of the LRT for OUN in bistable models leads also to a dependence of the type

$$T\sim\frac{\pi}{\alpha\sqrt{2}}e^{\alpha^2/4D\gamma}\left(1+\frac{3}{2}\alpha\tau+\cdots\right) \tag{4.11}$$

The coefficient 3/2 is the same appearing in the eigenvalue and mean first passage time analysis.[(29)] More details of all these calculations can be found in ref. 30.

5. CONCLUDING REMARKS

We have presented the exact solution of the general LRT problem for single-variable processes driven by dichotomous Markov noise and Poissonian white shot noise. We have emphasized the interest of the LRT in general as a characteristic time scale of the steady-state dynamics. The relative facility of getting explicit results for this quantity even for non-Markovian processes is of practical interest in order to check different approximations of the correlation functions and even to define new ones.[(10,11,19)] We have also compared our results with previous ones for Gaussian noises, and have given some examples of how to extract physical information in different prototype situations. Finally, from the formal properties of the general solutions of LRTs we have obtained some practical and theoretical conclusions. Among them we can stress the exact solution of some multiplicative-noise models and particularly the discussion about the characterization of the steady-state dynamics by means of an effective diffusion description in a Fokker–Planck scenario containing exact statics and exact LRTs. An extension of these technique to the study of transient dynamics has been done in the case of Gaussian noises in ref. 15 (white noise) and ref. 16 (colored noise) and is also possible for the non-Gaussian noises considered in this paper.[30]

APPENDIX. WHITE SHOT NOISE. GENERAL SOLUTION

In order apply the method of Section 2.1, the evolution operator of Eq. (2.37) can be written in a more convenient way as

$$L(x) = -\frac{\partial}{\partial x}\left[v(x) - \lambda w_0\, g(x)\right] - \lambda\left\{\left[1 - w_0 \frac{\partial}{\partial x}\, g(x)\right]^{-1} - 1\right\} \tag{A.1}$$

so that if we multiply (by the left) both members of (2.7) by the operator $1 - w_0(\partial/\partial x)\, g(x)$, Eq. (2.10) with (A.1) reduces to

$$\begin{aligned}\left\{\left[1 - w_0 \frac{\partial}{\partial x}\, g(x)\right] \frac{\partial}{\partial x}\left[v(x) - \lambda w_0\ \ g(x)\right] + \lambda w_0 \frac{\partial}{\partial x}\, g(x)\right\} \rho_0(x) \\ = \left[1 - w_0 \frac{\partial}{\partial x}\, g(x)\right]\left[f_2(x) - \langle f_2\rangle_{\text{st}}\right] P_{\text{st}}(x)\end{aligned} \tag{A.2}$$

This is the equation we have to solve now. A formal integration in both members leads to

$$\begin{aligned}\left\{v(x) + w_0\, g(x) \frac{\partial}{\partial x}\left[v(x) - \lambda w_0\, g(x)\right]\right\} \rho_0(x) \\ = \int_a^x \left[1 - w_0 \frac{\partial}{\partial x}\, g(x')\right]\left[f_2(x') - \langle f_2\rangle_{\text{st}}\right] P_{\text{st}}(x')\, dx' \\ \equiv -F_2(x) + w_0\, g(x)\, F_2'(x)\end{aligned} \tag{A.3}$$

The homogeneous part of (A.3) is the equation satisfied by $P_{\text{st}}(x)$; thus, the general solution of (A.3) is

$$\rho_0(x) = KP_{\text{st}}(x) + P_{\text{st}}(x) \int_a^x \frac{F_2(x') - w_0\, g(x')\, F_2'(x)}{w_0\, g(x')\left[\lambda w_0\, g(x') - v(x')\right)\right]\, P_{\text{st}}(x')}\, dx' \tag{A.4}$$

If we write the correlation function $C_{12}(s)$ in the equivalent form

$$C_{12}(s) = \int_a^b dx\, \left[f_1(x) - \langle f_1\rangle_{\text{st}}\right] e^{L(x)s}\left[f_2(x) - \langle f_2\rangle_{\text{st}}\right] P_{\text{st}}(x) \tag{A.5}$$

then Eq. (2.14) for $n=0$ reads

$$C_{12}^{(0)}=\int_a^b [f_1(x)-\langle f_1\rangle_{\rm st}]\,\rho_0(x)\,dx \tag{A.6}$$

so that the term proportional to K will not contribute.

Hence the LRT T_{12} will be given by (2.13) with the second term of the rhs of (A.4) substituted into (A.6). In the most important case $f_1=f_2=f$, an integration by parts in (A.6) with (A.4) gives

$$\int_a^b [f(x)-\langle f\rangle_{\rm st}]\,\rho_0(x)=\int_a^b \frac{F^2(x)-w_0\,g(x)\,F(x)\,F'(x)}{w_0\,g(x)[\lambda w_0\,g(x)-v(x)]\,P_{\rm st}(x)}\,dx \tag{A.7}$$

After another integration by parts in the second term of the integrand and some rearranging, Eq. (A.7) can be written as

$$TC(0)=\int_a^b \frac{F^2(x)}{[\lambda w_0\,g(x)-v(x)]\,P_{\rm st}(x)} \times\left(\frac{1}{w_0\,g(x)}+\frac{1}{2}\frac{d}{dx}\ln\{[v(x)-\lambda w_0\,g(x)]\,P_{\rm st}(x)\}\right)dx \tag{A.8}$$

Now, using the explicit form of the $P_{\rm st}(x)$,

$$P_{\rm st}(x)\propto\frac{1}{v(x)-\lambda w_0\,g(x)}\exp-\int^x\frac{v(x')}{w_0\,g(x')[v(x')-\lambda w_0\,g(x)]}\,dx' \tag{A.9}$$

Eq. (A.8) reduces to

$$T=\frac{1}{C(0)}\int_a^b \frac{F^2(x)}{w_0\,g(x)[\lambda w_0\,g(x)-v(x)]\,P_{\rm st}(x)}\left[1+\frac{1}{2}\frac{v(x)}{\lambda w_0\,g(x)-v(x)}\right]dx \tag{A.10}$$

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